weighted matchings in bipartite graphs

general problem: given complete bipartite graph with weights in $\mathbb{R}$ on edges, find the perfect matching of maximum weight $\left(w(M) = \sum_{e \in M} w(e) \right)$

Given G bipartite w/ weights on the edges, a <u>feasible vertex labeling</u> $\ell : V(G) \to \mathbb{R}$ is a function satisfying

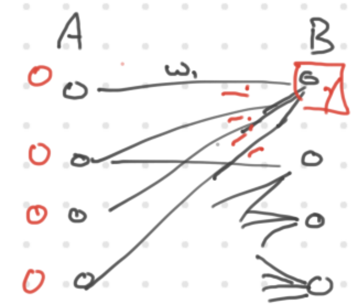
$$w(xy) \le \ell(x) + \ell(y)$$

weight of edge

vertex labeling satisfies sort of a Δ-inequality

<u>Obs</u> given G bipartite, $w: E(G) \to \mathbb{R}$, there always exists a feasible vertex labeling.

pf



$\ell = \max_{x \in A} w(xy)$

Let A, B be partition of $V(G)$

1) for every vertex $a \in A$, let $\ell(a) = 0$

2) for every vertex $b \in B$ let $\ell(b) = \max_{e \in \delta(b)} w(e)$

$\delta(b) :=$ edges incident to b.

Thm (Kuhn Munkes) G bipartite, $w: E(G) \to \mathbb{R}$
ℓ a feasible vertex labeling. Let H
be subgraph of edges $\{xy : \ell(x) + \ell(y) = w(xy)\}$
If M is a perfect matching in H, then M
is a maximum weight P.M. in G.

pf Fix G, w, & ℓ the labeling. &
Let H be the graph of edges xy
st $\ell(x) + \ell(y) = w(xy)$ & let M
be a p.m. in H.
Let M' be any other perfect matching
in G.
Consider $M' \Delta M$.

what does this look like.
more specifically, do $\exists$
vertices of deg 1 in
$M \Delta M'$?

if M & M'
overlap on
edge incident
to x, then
x has deg 0 in
$M \Delta M'$

if M & M'
don't overlap on
edge incident to x,

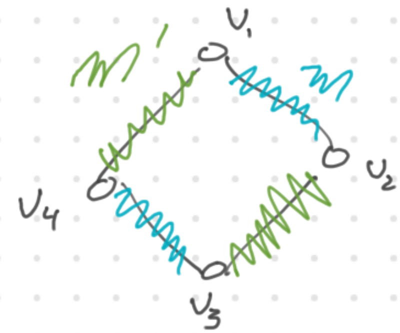
Then x will have deg 2 in $M \triangle M'$

but x must have deg 0 or 2 in $M \triangle M'$ because ~~The~~ x must be incident to an edge in both M & in M'.

$\Rightarrow$ $M \triangle M'$ has all deg $= 0$ or 2

$\Rightarrow$ $M \triangle M'$ is disjoint union of isolated vertices + cycles.

look at a cycle C - it must have even length because edges alternate between M & M'



label vertices $v_1, v_2, \ldots v_{2k}$ st $M \cap E(C)$ $v_1 v_2$ $v_3 v_4$ $v_5 v_6 \ldots$ $v_{2k-1} v_{2k}$

$M' \cap E(C) =$ $v_2 v_3, v_4 v_5, \ldots, v_{2k} v_1$

$$\omega(M' \cap E(C)) = \omega(v_2 v_3) + \omega(v_4 v_5) + \cdots + \omega(v_{2k} v_1)$$

$$\leq \ell(v_2) + \ell(v_3) + \ell(v_4) + \ell(v_5) + \cdots + \ell(v_{2k}) + \ell(v_1)$$

$$= \omega(v_1 v_2) + \omega(v_3 v_4) + \cdots + \omega(v_{2k-1} v_{2k})$$

all edges of M are in H where $\omega(xy) = \ell(x) + \ell(y)$

$= w(M)$

conclusion is that

$$w(M' \cap C) \le w(M \cap C)$$

& this holds for all cycles C in $M' \Delta M$.

$$\Rightarrow w(M' \setminus M) \le w(M \setminus M')$$

$$\Rightarrow w(M) \ge w(M')$$

& since this holds for any perfect matching M', we see that M is a maximum weight matching.

algorithm starts w/ a feasible labeling & matching M in the subgraph H of edges s.t. $w(xy) = \ell(x) + \ell(y)$

Hungarian method

input: bipartite G w/ weights w

Find a feasible labeling ℓ

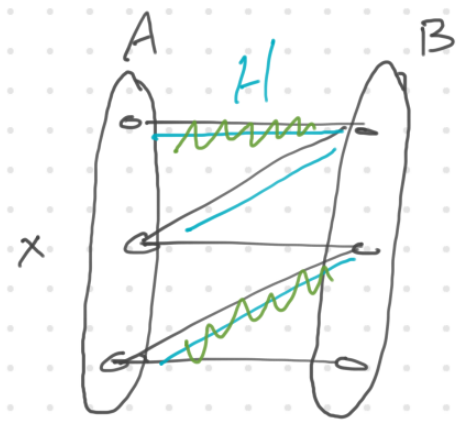
Find equality graph H

Find a matching M in H

While M is not perfect Do

- Fix x an unmatched vertex in A
- Grow M-alternating BFS tree T rooted at x in H
- While $\nexists$ an M-augmenting path in T Do
  - let $X = A \cap V(T)$, $Y = B \cap V(T)$
  - cl 1 $\Delta = \min_{u \in X, v \in B \setminus Y} \ell(u) + \ell(v) - w(uv)$
  - cl 2 $\ell(v) = \begin{cases} \ell(v) - \Delta : & v \in X \\ \ell(v) + \Delta : & v \in Y \\ \ell(v) & \text{otherwise} \end{cases}$
  - cl 3 H new equality graph, T new BFS tree rooted at x
- let P be an M-augmenting path
- $M = M \Delta P$

return M



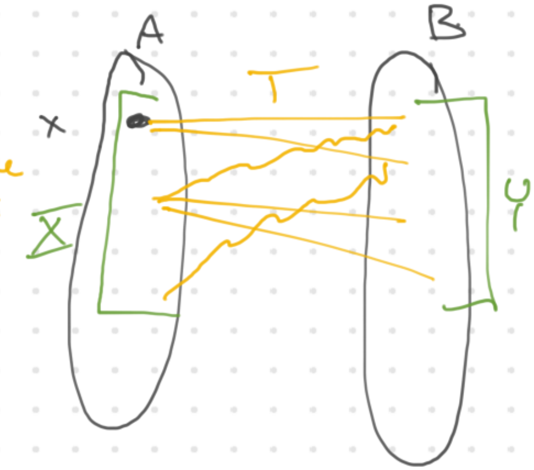
G bipartite w/ partition (A, B)

H set of edges xy w/ $\ell(x) + \ell(y) = w(xy)$

M a matching in H

Fix to be an unmatched vertex

T BFS Tree rooted at x



If algorithm terminates, Then we have a perfect matching M + feasible labeling ℓ ST $M \subseteq$ set of edges $xy : \ell(x) + \ell(y) = w(xy)$

$\Rightarrow$ Thm says M is a max weight matching.

To prove correctness, we just need to show That the algorithm terminates

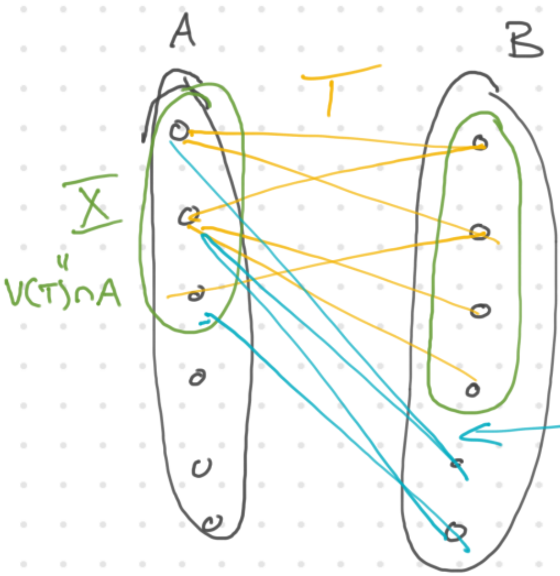
The outer while loop will always terminate because our matching M is getting larger with each pass, and remember, we're in a complete bipartite graph, so it will terminate with a p.m.

So in order to prove That alg. terminates, it suffices to show The inner while loop always terminates.

To prove The inner while loop always terminates we prove 3 claims

Cl 1 $\quad \Delta > 0$

pf since $w(xy) \le \ell(x) + \ell(y)$ we just need to prove that no possible edge in the definition of Δ satisfies $w(xy) = \ell(x) + \ell(y)$



$V(T) \cap B$

T = BFS alternating tree

edges uv used Δ

$\{uv \in E(G) : u \in \overline{X},\ v \in B \setminus Y\}$

cl is equivalent to saying that no blue edge is in H

Observe that no edge from $\overline{X} \to B \setminus Y$ is a matching edge

$\overline{X}$ & Y were defined by BFS M-alternating tree + every matching touching a vertex of T is contained in T

But

but blue edges go from $\overline{X} \subseteq V(T)$ to a vertex ~~$Y\setminus I$~~ $B \setminus Y$ which is disjoint from T $\Rightarrow$ no blue edge is in $\mathcal{M}$.

Since the layers of T alternate between A + B, every time we get to an A-vertex we ~~are at~~ add all possible edges from there vertex to the tree. Conclusion is, if there were a blue edge in H, call it uv w/ $u \in \overline{X}$, $v \in B \setminus Y$, the tree T, upon reaching vertex u, would have added the edge uv to T.

By construction of T, no blue edge was available to continue on for each vertex in $\overline{X} = V(T) \cap A$
$\Rightarrow$ No blue edge is in H.

Conclusion: uv w/ $u \in \overline{X}, v \in B \setminus Y$ $\notin E(H) \Rightarrow w(uv) < \ell(u) + \ell(v)$

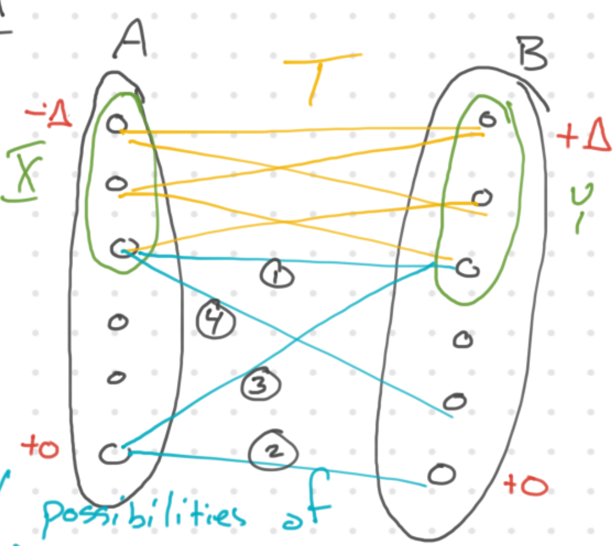
$$\Rightarrow \cancel{\min}\ \Delta = \min_{u \in \overline{X},\, v \in B\setminus Y} \ell(u) + \ell(v) - w(uv) > 0$$ as claimed.

Cl 2 new ℓ defined as

$$\ell(x) = \begin{cases} \ell(x) - \Delta & x \in \overline{X} \\ \ell(x) + \Delta & x \in Y \\ \ell(x) & \text{otherwise} \end{cases}$$

is a feasible labeling

Pf



4 possibilities of edges & we need to check Δ-ineq. still holds

① edges from $\overline{X} \to Y$
add Δ to one end & subtract Δ from other, so Δ-ineq still holds

② edges from $A \setminus \overline{X} \to B \setminus Y$
same as ①

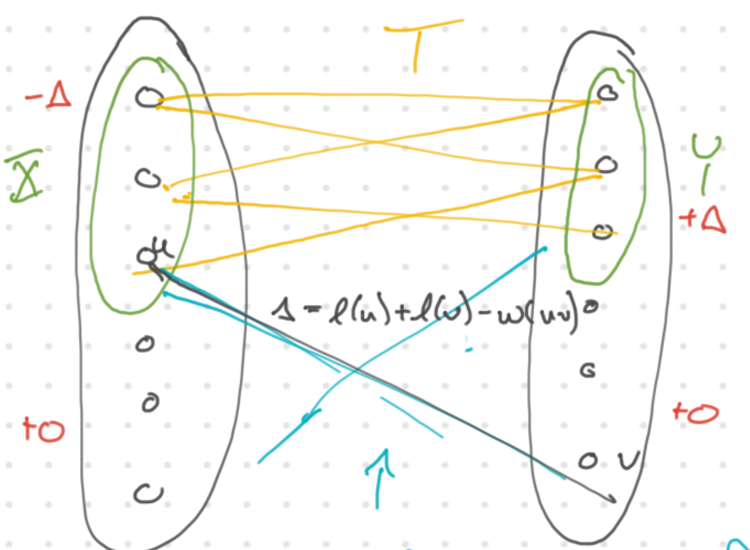
③ edges from $A \setminus \overline{X}$ to Y
added an extra Δ to $\ell(u) + \ell(v)$
so we still satisfy $\ell(u) + \ell(v) \geq w(uv)$

④ edges from $\overline{X} \to B \setminus Y$
but our choice of Δ ensures that
$\ell(u) + \ell(v) - \Delta \geq w(uv)$ for all edges uv w/ $u \in \overline{X}$, $v \in B \setminus Y$, so Δ-ineq is satisfied.

Cl 3 H_{new} be the ~~set o~~ subgraph of edges ~~of~~ uv w/ $\ell(u)+\ell(v)=w(uv)$

$\mathcal{M}$ is a subgraph of H_{new} +

$T_{new} :=$ The BFS $\mathcal{M}$-alternating tree with root x is strictly larger than T



we proved no $\mathcal{M}$ edge from $\bar{X} \cup Y \to V(G) \setminus (\bar{X} \cup Y)$

we already proved no $\mathcal{M}$-edge goes from $\bar{X} \cup Y$ to $V(G) \setminus (\bar{X} \cup Y)$ because all $\mathcal{M}$ edges touching T are in T

$\Rightarrow$ every edge of $\mathcal{M}$ ~~got~~ either got $+\Delta$ on one end and $-\Delta$ other or it's ℓ values stayed the same

$\Rightarrow$ every edge of $\mathcal{M}$ is in H_{new}

+ for same reason, all edges of T are also in H_{new} $\Rightarrow$ BFS $\mathcal{M}$-alternating tree in H_{new} contains T

& $\exists$ at least one edge uv w/ $u \in \overline{X}$, $v \in B \setminus Y$ st $\ell(u) + \ell(v) - \omega(uv) = \Delta$
$\Rightarrow$ $uv \in H_{new}$ & when BFS M-alternating tree gets to u, the edge uv is added to the tree.

$\Rightarrow$ T_{new} is strictly larger than T

Conclusion from 1, 2, 3, we keep growing T at each pass until eventually, T contains an M-augmenting path, ie inner while loop terminates.

Thus the alg terminates, & theorem implies that the final matching is a max. weight perfect matching.